INTEGRAL OF CALCULUS

Sigma and Limit Notation

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Contents

Sigma Notation (1)

The Greek letter $\sum$ (pronounced *sigma*), which means the sum of, is generally used to express a series in a concise way.

example

$$2+4+8+16+32=2^1+2^2+2^3+2^4+2^5$$

$$=\sum_{n=1}^{5} 2^n$$

Sigma Notation (2)

- As general,

$$\sum_{n=c}^{m} U_n$$

- The numbers at the top and bottom of sigma are called boundaries and tell us what numbers we substitute in to the expression for the terms in our sum. What comes after sigma is an algebraic expression representing terms in the sum.

  In the symbolic notation above, *n* is a variable and U_n represents the terms in our sum.

Sigma Notation (3)

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots \quad (1)$$

$$5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 17,\ 19,\ \ldots \quad (2)$$

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ \ldots \quad (3)$$

$$1,\ \sqrt{2},\ \sqrt{3},\ 2,\ \sqrt{5},\ \sqrt{6},\ \sqrt{7},\ 2\sqrt{2},\ \ldots \quad (4)$$

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \frac{5}{6},\ \frac{6}{7},\ \frac{7}{8},\ \frac{8}{9},\ \ldots \quad (5)$$

$$2,\ 0,\ -2,\ -4,\ -6,\ -8,\ -10,\ -12,\ \ldots \quad (6)$$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \ldots \quad (7)$$

$$1,\ \frac{1}{2},\ 3,\ \frac{1}{4},\ 5,\ \frac{1}{6},\ 7,\ \frac{1}{8},\ \ldots \quad (8)$$

$$1,\ -1,\ \frac{1}{3},\ -\frac{1}{3},\ \frac{1}{5},\ -\frac{1}{5},\ \frac{1}{7},\ -\frac{1}{7}\ \ldots \quad (9)$$

$$1,\ \frac{2}{3},\ \frac{1}{3},\ \frac{4}{4},\ \frac{1}{5},\ \frac{6}{7},\ \frac{1}{7},\ \frac{8}{9},\ \ldots \quad (10)$$

Sigma Notation (4)

- In each example, we have eight terms of a sequence. Could you write, say, the ninth term?
- Why? Could you tell me the certain law such that you could recognize the ninth term is your answer.

Let us look at the situation more rigorously. Let us enumerate all the terms of the sequence in sequential order i.e 1, 2, 3, ... , n, ... There is a certain law by which each of these natural numbers is assigned to a certain number. For example, in example (1), this arrangement is as follows :

$$\begin{array}{cccccccccl} 1 & 2 & 4 & 8 & 16 & 32 & \ldots & 2^{n-1} & \ldots & \text{(terms of the sequence)} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & & \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots & n & \ldots & \text{(position numbers of the terms)} \end{array}$$

Sigma Notation (5)
Write the certain law

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots \tag{1}$$

$$5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 17,\ 19,\ \ldots \tag{2}$$

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ \ldots \tag{3}$$

$$1,\ \sqrt{2},\ \sqrt{3},\ 2,\ \sqrt{5},\ \sqrt{6},\ \sqrt{7},\ 2\sqrt{2},\ \ldots \tag{4}$$

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \frac{5}{6},\ \frac{6}{7},\ \frac{7}{8},\ \frac{8}{9},\ \ldots \tag{5}$$

$$2,\ 0,\ -2,\ -4,\ -6,\ -8,\ -10,\ -12,\ \ldots \tag{6}$$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \ldots \tag{7}$$

$$1,\ \frac{1}{2},\ 3,\ \frac{1}{4},\ 5,\ \frac{1}{6},\ 7,\ \frac{1}{8},\ \ldots \tag{8}$$

$$1,\ -1,\ \frac{1}{3},\ -\frac{1}{3},\ \frac{1}{5},\ -\frac{1}{5},\ \frac{1}{7},\ -\frac{1}{7}\ \ldots \tag{9}$$

$$1,\ \frac{2}{3},\ \frac{1}{3},\ \frac{4}{4},\ \frac{1}{5},\ \frac{6}{7},\ \frac{1}{7},\ \frac{8}{9},\ \ldots \tag{10}$$

Sigma Notation (6)
Series

- 1,2,4,8,16,32,64, 128, ... $U_n = 2^{n-1}$
- 5,7,9,11,13,15,17,19, ... $U_n = 3 + 2n$
- 1,4,9,16,25,36,49,64, ... $U_n = n^2$

dst..

- 1+2+4+8+16+32+64+128 $S_8 = \sum_{n=1}^{8} 2^{n-1}$
- 5+7+9+11+13+15+17+19 $S_8 = \sum_{n=1}^{8} 3 + 2n$
- 1+4+9+16+25+36+49+64 $S_8 = \sum_{n=1}^{8} n^2$

Sigma Notation (7)
Series for integration

- 1,2,3,4,5,... Un ?   Sn ?

$$U_n = n, S_m = \sum_{n=1}^{m} n = \frac{m(m+1)}{2}$$

- 1,4,9,16,25,... Un? Sn?

$$U_n = n^2, S_m = \sum_{n=1}^{m} n^2 = \frac{m(m+1)(2m+1)}{6}$$

- 1,8,27,64,125,... Un? Sn?

$$U_n = n^3, S_m = \sum_{n=1}^{m} n^3 = \left[\frac{m(m+1)}{2}\right]^2$$

Sigma Notation (8)
Linearity of Σ

- If it's given $\{a_n\}$ and $\{b_n\}$ two sequences and c is a constant, then

$$(1)\sum_{n=1}^{m} ca_n = c\sum_{n=1}^{m} a_n$$

$$(2)\sum_{n=1}^{m} (a_n + b_n) = \sum_{n=1}^{m} a_n + \sum_{n=1}^{m} b_n$$

$$(3)\sum_{n=1}^{m} (a_n - b_n) = \sum_{n=1}^{m} a_n - \sum_{n=1}^{m} b_n$$

Sigma Notation (9)
Change the boundaries

- This value of both sigma is equivalent.

$$\sum_{n=c}^{m} U_n = \sum_{n=0}^{m-c} U_{n+c}$$

$$\sum_{n=c}^{m} U_n = \sum_{i=c}^{m} U_i$$

Example and Exercises (1)

- Find the value of

$$\sum_{n=1}^{100}\left(n^2+2n+1\right)=\ldots.$$

- Answer

$$\sum_{n=1}^{100}\left(n^2+2n+1\right)=\sum_{n=1}^{100}n^2+\sum_{n=1}^{100}2n+\sum_{n=1}^{100}1=\sum_{n=1}^{100}n^2+2\sum_{n=1}^{100}n+\sum_{n=1}^{100}1$$

$$=\frac{100(100+1)(2.100+1)}{6}+2\left(\frac{100(100+1)}{2}\right)+100=348.550$$

Example and Exercises (2)

Find

$$1.\sum_{n=1}^{100}(3n-2)=\ldots.$$

$$4.\sum_{n=4}^{100}n^2=\ldots.$$

$$2.\sum_{k=1}^{10}\left(k^3-k^2\right)=\ldots.$$

$$5.\sum_{k=0}^{10}k^3=\ldots.$$

$$3.\sum_{i=1}^{10}[(i-1)(4i+3)]=\ldots.$$

$$6.\sum_{i=5}^{10}[i+10]^2=\ldots.$$

Limit Notation (1)
Definition

- The limit of a function (if it exists) for some x-value, a, is the height the function gets closer and closer to as x gets closer and closer to a from the left and the right.

Limit Notation (2)
Limit for Integration

As general,

$$\lim_{x \to a} f(x)$$

For integration

$$\lim_{n \to \infty} A(R_n)$$

Limit Notation (3)
Properties Needed

If *f(n)* and *g(n)* are polinomial degree p and q, respectively then

a. For p < q then $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$

b. For p > q then $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

c. For p = q then $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \frac{koef\ \ n^p}{koef\ \ n^q}$

Examples and Exercises (1)

Find the limit value $\lim_{n\to\infty} A(R_n)$ for

$$A(R_n) = \frac{4}{3} - \frac{5}{n} + \frac{6}{3n^2}$$

Find the limit value $\lim_{n\to\infty} A(S_n)$ for

$$A(S_n) = \frac{7}{2} + \frac{4}{3n^2}$$

Examples and Exercises (2)

$$\lim_{n\to\infty} A(R_n) = \lim_{n\to\infty}\left(\frac{4}{3} - \frac{5}{n} + \frac{6}{3n^2}\right)$$

$$= \lim_{n\to\infty}\frac{4}{3} - \lim_{n\to\infty}\frac{5}{n} + \lim_{n\to\infty}\frac{6}{3n^2} = \frac{4}{3} - 0 + 0 = \frac{4}{3}$$

$$\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty}\left(\frac{7}{2} + \frac{4}{3n^2}\right)$$

$$= \lim_{n\to\infty}\frac{7}{2} + \lim_{n\to\infty}\frac{4}{3n^2} = \frac{7}{2} + 0 = \frac{7}{2}$$

Examples and Exercises (3)

1. It is given that $A(R_n) = \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$. Find the limit value of

$$\lim_{n\to\infty} A(R_n)$$

2. It is given that $A(S_n) = \frac{81}{16}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) + 3$ Find the limit value of

$$\lim_{n\to\infty} A(S_n)$$

References

- Varberg, Dale. Purcell, Edwin J. Calculus. Seventh Edition.
- Tarasov. Calculus. MIR Publisher. Moscow.